

Exercise 1

February 16, 2010

Due one week later. Answers to selected problems will be posted.

1) Represent a sequence over the alphabet H, T as a Hidden Markov chain with states E and M, with parameters as follows: initial probabilities $\{\pi_E = \frac{1}{5}, \pi_M = \frac{4}{5}\}$, transition probabilities $\{\tau_{EE} = \frac{1}{2}, \tau_{EM} = \frac{1}{2}, \tau_{ME} = \frac{1}{4}, \tau_{MM} = \frac{3}{4}\}$, and output probabilities $\{P(H|E) = \frac{1}{3}, P(T|E) = \frac{2}{3}, P(H|M) = \frac{1}{6}, P(T|M) = \frac{5}{6}\}$.

a) Calculate $P_\lambda(Q_2 = M|HTH)$ using the forward and backward algorithms.

b) Check your answer in a) using the total probability method.

Solution: a)

$$\begin{aligned} P(Q_2 = M|HTH) &= \frac{P(Q_2=M, O_1=H, O_2=T, O_3=H)}{P_\lambda(HTH)} \\ &= \frac{P(O_1 O_2=HT, Q_2=M) \cdot P(O_3=H|Q_2=M)}{P_\lambda(HTH)} \end{aligned}$$

The forward algorithm:

$$E_1 = \pi_E \cdot P(H|E) = \frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$$

$$M_1 = \pi_M \cdot P(H|M) = \frac{4}{5} \cdot \frac{1}{6} = \frac{2}{15}$$

$$E_2 = [E_1 \cdot \tau_{EE} + M_1 \cdot \tau_{ME}] \cdot P(T|E) = [\frac{1}{15} \cdot \frac{1}{2} + \frac{2}{15} \cdot \frac{1}{4}] \cdot \frac{2}{3} = \frac{2}{45}$$

$$M_2 = [E_1 \cdot \tau_{EM} + M_1 \cdot \tau_{MM}] \cdot P(T|M) = [\frac{1}{15} \cdot \frac{1}{2} + \frac{2}{15} \cdot \frac{3}{4}] \cdot \frac{5}{6} = \frac{1}{9}$$

$$E_3 = [\frac{2}{45} \cdot \frac{1}{2} + \frac{1}{9} \cdot \frac{1}{4}] \cdot \frac{1}{3} = \frac{1}{60}$$

$$M_3 = [\frac{2}{45} \cdot \frac{1}{2} + \frac{1}{9} \cdot \frac{3}{4}] \cdot \frac{1}{6} = \frac{19}{1080}$$

The backward algorithm:

$$P(O_3 = H|Q_2 = M) = m_2 = P(H|M) \cdot \tau_{MM} \cdot m_3 + P(H|E) \cdot \tau_{ME} \cdot e_3$$

Here, $N = 3$, so $e_3 = m_3 = 1$. Thus,

$$m_2 = \frac{1}{6} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{5}{24}$$

Use:

$$P_\lambda(HTH) = E_3 + M_3 = \frac{37}{1080}$$

$$\text{Thus, } P(Q_2 = M|HTH) = \frac{M_2 \cdot m_2}{P_\lambda(HTH)} = \frac{\frac{1}{9} \cdot \frac{5}{24}}{\frac{37}{1080}} = \frac{25}{37}$$

Solution: b)

$P(O_1O_2 Q_1Q_2)$	EEE	EEM	EME	EMM	MEE	MEM	MME	MMM
HHH	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{6}$	$\frac{1}{3} \cdot \frac{1}{6} \cdot \frac{1}{3}$	$\frac{1}{3} \cdot \frac{1}{6} \cdot \frac{1}{6}$	$\frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{3}$	$\frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{6}$	$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{3}$	$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$
HHT	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{5}{6}$	$\frac{1}{3} \cdot \frac{1}{6} \cdot \frac{2}{3}$	$\frac{1}{3} \cdot \frac{1}{6} \cdot \frac{5}{6}$	$\frac{1}{6} \cdot \frac{1}{3} \cdot \frac{2}{3}$	$\frac{1}{6} \cdot \frac{1}{3} \cdot \frac{5}{6}$	$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{2}{3}$	$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6}$
HTH	$\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}$	$\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{6}$	$\frac{1}{3} \cdot \frac{5}{6} \cdot \frac{1}{3}$	$\frac{1}{3} \cdot \frac{5}{6} \cdot \frac{1}{6}$	$\frac{1}{6} \cdot \frac{2}{3} \cdot \frac{1}{3}$	$\frac{1}{6} \cdot \frac{2}{3} \cdot \frac{1}{6}$	$\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{3}$	$\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$
HTT	$\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$	$\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{6}$	$\frac{1}{3} \cdot \frac{5}{6} \cdot \frac{2}{3}$	$\frac{1}{3} \cdot \frac{5}{6} \cdot \frac{5}{6}$	$\frac{1}{6} \cdot \frac{2}{3} \cdot \frac{2}{3}$	$\frac{1}{6} \cdot \frac{2}{3} \cdot \frac{5}{6}$	$\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{2}{3}$	$\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$
THH	$\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$	$\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{6}$	$\frac{2}{3} \cdot \frac{1}{6} \cdot \frac{1}{3}$	$\frac{2}{3} \cdot \frac{1}{6} \cdot \frac{1}{6}$	$\frac{2}{6} \cdot \frac{1}{3} \cdot \frac{1}{3}$	$\frac{2}{6} \cdot \frac{1}{3} \cdot \frac{1}{6}$	$\frac{2}{6} \cdot \frac{1}{6} \cdot \frac{1}{3}$	$\frac{2}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$
THT	$\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$	$\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{5}{6}$	$\frac{2}{3} \cdot \frac{1}{6} \cdot \frac{2}{3}$	$\frac{2}{3} \cdot \frac{1}{6} \cdot \frac{5}{6}$	$\frac{2}{6} \cdot \frac{1}{3} \cdot \frac{2}{3}$	$\frac{2}{6} \cdot \frac{1}{3} \cdot \frac{5}{6}$	$\frac{2}{6} \cdot \frac{1}{6} \cdot \frac{2}{3}$	$\frac{2}{6} \cdot \frac{1}{6} \cdot \frac{5}{6}$
TTH	$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}$	$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{6}$	$\frac{2}{3} \cdot \frac{5}{6} \cdot \frac{1}{3}$	$\frac{2}{3} \cdot \frac{5}{6} \cdot \frac{1}{6}$	$\frac{2}{6} \cdot \frac{2}{3} \cdot \frac{1}{3}$	$\frac{2}{6} \cdot \frac{2}{3} \cdot \frac{1}{6}$	$\frac{2}{6} \cdot \frac{5}{6} \cdot \frac{1}{3}$	$\frac{2}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$
TTT	$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$	$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{5}{6}$	$\frac{2}{3} \cdot \frac{5}{6} \cdot \frac{2}{3}$	$\frac{2}{3} \cdot \frac{5}{6} \cdot \frac{5}{6}$	$\frac{2}{6} \cdot \frac{2}{3} \cdot \frac{2}{3}$	$\frac{2}{6} \cdot \frac{2}{3} \cdot \frac{5}{6}$	$\frac{2}{6} \cdot \frac{5}{6} \cdot \frac{2}{3}$	$\frac{2}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$
$P(Q_1Q_2)$	$\frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{4}$	$\frac{1}{5} \cdot \frac{1}{2} \cdot \frac{3}{4}$	$\frac{4}{5} \cdot \frac{1}{4} \cdot \frac{1}{2}$	$\frac{4}{5} \cdot \frac{1}{4} \cdot \frac{1}{2}$	$\frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{4}$	$\frac{4}{5} \cdot \frac{3}{4} \cdot \frac{3}{4}$
Total	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{3}{40}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{9}{20}$

$$\begin{aligned}
 P(Q_2 = M | HTH) &= \frac{P(Q_2=M \text{ and } HTH)}{P(HTH)} \\
 &= \frac{\frac{5}{54} \cdot \frac{1}{40} + \frac{5}{108} \cdot \frac{3}{40} + \frac{5}{108} \cdot \frac{3}{20} + \frac{5}{216} \cdot \frac{9}{20}}{\frac{2}{27} \cdot \frac{1}{20} + \frac{1}{27} \cdot \frac{1}{20} + \frac{5}{54} \cdot \frac{1}{40} + \frac{5}{108} \cdot \frac{3}{40} + \frac{1}{27} \cdot \frac{1}{10} + \frac{1}{54} \cdot \frac{1}{10} + \frac{5}{108} \cdot \frac{3}{20} + \frac{5}{216} \cdot \frac{9}{20}} \\
 &= \frac{\frac{5}{216}}{\frac{37}{1080}} \\
 &= \frac{25}{37}
 \end{aligned}$$

Credit to Susan VanderPlas and Rani Walia.