BCB BCB/GDCB/STAT/COM S 568 Spring 2010

Homework 3 Solution

February 2, 2010

Due one week later. Answers to selected problems will be posted.

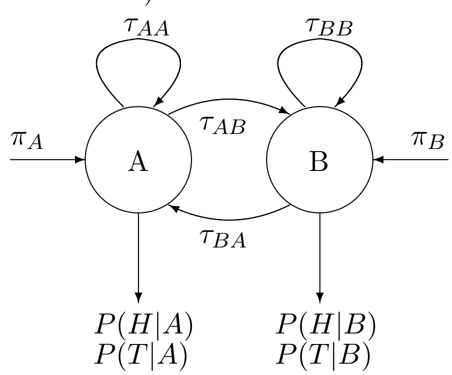
1) Suppose we have two coins (A and B). Coin A has probability 0.7 for coming up heads. Coin B has probability 0.4 for coming up heads. If we use coin A for a given toss, we will retain it for the next toss with probability 0.4. If we use coin B for a given toss, the probability to retain coin B for the next toss is 0.8. The coin initially flipped is equally likely to be coin A or coin B.

a) Formulate the experiment as a Hidden Markov Model. State all parameters of the model.

b) What is the probability to get "HT" on two successive tosses (H: Head; T: Tail)? Calculate the probability by complete enumeration of possibilities and also by the HMM "forward" algorithm. Compare your answers.

c) If we get "HT", what is the most likely sequence of coins (AA, AB, BA, or BB)? Calculate the probability by complete enumeration of possibilities and also by the HMM "Viterbi" algorithm. Compare your answers.

Solution: a)



$$\begin{cases} \pi_A &= 0.5 \\ \pi_B &= 0.5 \end{cases} \qquad \begin{cases} \tau_{AA} &= 0.4 \\ \tau_{AB} &= 0.6 \\ \tau_{BA} &= 0.2 \\ \tau_{BB} &= 0.8 \end{cases} \qquad \begin{cases} P(H|A) &= 0.7 \\ P(T|A) &= 0.3 \\ P(H|B) &= 0.4 \\ P(T|B) &= 0.6 \end{cases}$$

Solution: b)

	AA	AB	BA	BB
HH	0.7×0.7	0.7×0.4	0.4×0.7	0.4×0.4
HT	0.7×0.3	0.7×0.6	0.4×0.3	0.4×0.6
TH	0.3×0.7	0.3×0.4	0.6×0.7	0.6×0.4
TT	0.3×0.3	0.3×0.6	0.6×0.3	0.6×0.6
Prob$\{Q_1,Q_2\}$	0.5×0.4	0.5×0.6	0.5×0.2	0.5×0.8

By complete enumeration of possibilities.

$$\begin{aligned}\text{Prob}\{\text{HT}\} &= (0.7\times0.3)\times(0.5\times0.4)+(0.7\times0.6)\times(0.5\times0.6)+(0.4\times0.3)\times(0.5\times0.2)+(0.4\times0.6)\times(0.5\times0.8) \\ &= 0.042+0.126+0.012+0.096 = 0.276\end{aligned}$$

By the HMM "forward" algorithm.

$A_1 = \pi_A \times \text{P}(O_1\,|\text{A}) = 0.5\times0.7 = 0.35$
$B_1 = \pi_B \times \text{P}(O_1\,|\text{B}) = 0.5\times0.4 = 0.2$
$A_2 = [A_1 \times \tau_{AA} + B_1 \times \tau_{BA}]\times \text{P}(O_2\,|\text{A}) = (0.35\times0.4+0.2\times0.2)\times0.3 = 0.054$
$B_2 = [A_1 \times \tau_{AB} + B_1 \times \tau_{BB}]\times \text{P}(O_2\,|\text{B}) = (0.35\times0.6+0.2\times0.8)\times0.6 = 0.222$
$\text{Prob}\{\text{HT}\} = A_2 + B_2 = 0.054 + 0.222 = 0.276$
The results from those two different methods are the same.

Solution: c)
By complete enumeration of possibilities.

From the results of 1-b), we know that the probabilities to get "HT" with coin sequence AA, AB, BA, or BB are 0.042, 0.126, 0.012, and 0.096, respectively. Therefore, the most likely sequence of coins should be AB, which has the highest probability, 0.126.

By the HMM "Viterbi" algorithm.

$A_1 = \pi_A \times \text{P}(O_1\,|\text{A}) = 0.5\times0.7 = 0.35$
$B_1 = \pi_B \times \text{P}(O_1\,|\text{B}) = 0.5\times0.4 = 0.2$
$A_2 = \max[A_1 \times \tau_{AA},\, B_1 \times \tau_{BA}]\times \text{P}(O_2\,|\text{A}) = \max(0.35\times0.4,\, 0.2\times0.2)\times0.3 = 0.14\times0.3 = 0.042$
$B_2 = \max[A_1 \times \tau_{AB},\, B_1 \times \tau_{BB}]\times \text{P}(O_2\,|\text{B}) = \max(0.35\times0.6,\, 0.2\times0.8)\times0.6 = 0.21\times0.6 = 0.126$
$\max_{\{Q\}} P_\lambda(\text{Q, O}) = \max(A_2,\, B_2) = 0.126$

The maximum 0.126 comes from B_2, and B_2 derives from A_1 ("back tracing"). So the most likely sequence of coins should be AB.
The results from those two different methods ("complete enumeration of possibilities" and "the HMM Viterbi algorithm") are the same.

2) Solve the problems posted in (1) for the word "TT" instead of "HT".

Solution: b)

	AA	AB	BA	BB
HH	0.7×0.7	0.7×0.4	0.4×0.7	0.4×0.4
HT	0.7×0.3	0.7×0.6	0.4×0.3	0.4×0.6
TH	0.3×0.7	0.3×0.4	0.6×0.7	0.6×0.4
TT	0.3×0.3	0.3×0.6	0.6×0.3	0.6×0.6
Prob$\{Q_1,Q_2\}$	0.5×0.4	0.5×0.6	0.5×0.2	0.5×0.8

By complete enumeration of possibilities.

$$\text{Prob}\{\text{TT}\} = (0.3\times0.3)\times(0.5\times0.4)+(0.3\times0.6)\times(0.5\times0.6)+(0.6\times0.3)\times(0.5\times0.2)+(0.6\times0.6)\times(0.5\times0.8)$$
$$= 0.018+0.054+0.018+0.144 = 0.234$$

By the HMM "forward" algorithm.

$A_1 = \pi_A\times\text{P}(O_1\,|\text{A}) = 0.5\times0.3 = 0.15$
$B_1 = \pi_B\times\text{P}(O_1\,|\text{B}) = 0.5\times0.6 = 0.3$
$A_2 = [A_1\times\tau_{AA} + B_1\times\tau_{BA}]\times\text{P}(O_2\,|\text{A}) = (0.15\times0.4+0.3\times0.2)\times0.3 = 0.036$
$B_2 = [A_1\times\tau_{AB} + B_1\times\tau_{BB}]\times\text{P}(O_2\,|\text{B}) = (0.15\times0.6+0.3\times0.8)\times0.6 = 0.198$
$\text{Prob}\{\text{TT}\} = A_2 + B_2 = 0.054 + 0.222 = 0.234$

The results from those two different methods are the same.

Solution: c)

By complete enumeration of possibilities.

From the results of 2-b), we know that the probabilities to get "TT" with coin sequence AA, AB, BA, or BB are 0.018, 0.054, 0.018, and 0.144, respectively. Therefore, the most likely sequence of coins should be BB, which has the highest probability, 0.144.

By the HMM "Viterbi" algorithm.

$A_1 = \pi_A\times\text{P}(O_1\,|\text{A}) = 0.5\times0.3 = 0.15$
$B_1 = \pi_B\times\text{P}(O_1\,|\text{B}) = 0.5\times0.6 = 0.3$
$A_2 = \max[A_1\times\tau_{AA},\, B_1\times\tau_{BA}]\times\text{P}(O_2\,|\text{A}) = \max(0.15\times0.4,\, 0.3\times0.2)\times0.3 = 0.06\times0.3 = 0.018$
$B_2 = \max[A_1\times\tau_{AB},\, B_1\times\tau_{BB}]\times\text{P}(O_2\,|\text{B}) = \max(0.15\times0.6,\, 0.3\times0.8)\times0.6 = 0.24\times0.6 = 0.144$
$\max_{\{Q\}}P_\lambda(\text{Q},\,\text{O}) = \max(A_2,\,B_2) = 0.144$

The maximum 0.126 comes from B_2, and B_2 derives from B_1. So the most likely sequence of coins should be BB.
The results from those two different methods ("complete enumeration of possibilities" and "the HMM Viterbi algorithm") are the same.